

# PHIL 4310: Advanced Logic

Spring 2026 — Homework 3

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**Reading:** Finish reading Chapter 2 of *Logic for Philosophy*.

## Part I The Soundness Theorem

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1. Show that the  $\rightarrow$ E rule is validity preserving.
2. Show that the  $\rightarrow$ I rule is validity preserving.
3. Imagine that we added rules to Sider's sequent system. Would the Soundness Theorem continue to hold if we added the following rules? (*These are two separate questions—the rules are not both being added simultaneously.*) Explain your answers.

$$\frac{\Delta \Rightarrow \varphi \rightarrow \psi \quad \Sigma \Rightarrow \chi \rightarrow \psi}{\Delta, \Sigma \Rightarrow (\varphi \vee \chi) \rightarrow \psi} (\vee \rightarrow \text{Intro})$$

$$\frac{\Delta \Rightarrow (\varphi \leftrightarrow \psi) \rightarrow \chi \quad \Sigma \Rightarrow \varphi \rightarrow \chi}{\Delta, \Sigma \Rightarrow \psi \rightarrow \chi} (\leftrightarrow \rightarrow \text{Elim})$$

4. Unsound rules are very bad. Imagine that we thought that  $\vee$  worked like 'exactly one' instead of 'at least one'. We might then think that the following rule would be acceptable:

$$\frac{\Delta \Rightarrow \varphi \quad \Sigma \Rightarrow \varphi \vee \psi}{\Delta, \Sigma \Rightarrow \sim \psi} (\text{Ex-}\vee\text{Elim})$$

Show that if we add this rule to Sider's proof system, we can derive  $P \Rightarrow \sim P$ .

*Extra credit:* In fact, adding this rule allows us to prove *any* possible sequent. Prove that this is true.

*Extra, extra credit:* Moreover, adding *any* unsound rule with metavariables in the style of the other Sider rules makes every possible sequent derivable. Prove this.

5. **More on Soundness and Completeness.** Prove that for any formula  $\varphi$ , *exactly one* of the following three conditions holds:

- (a)  $\{ \} \Rightarrow \varphi$  is a **derivable** sequent.
- (b)  $\{ \} \Rightarrow \sim\varphi$  is a **derivable** sequent.
- (c) There is a PL interpretation where  $V(\varphi) = 1$  *and* a PL interpretation where  $V(\sim\varphi) = 1$ .

For this problem you may assume without argument that both the Soundness Theorem and the Completeness Theorem hold for PL. If you use either theorem, indicate clearly *which* theorem you are using and *exactly how* you are using it in your proof.

## Part II Consequences of a Valid Sequent

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Assume that  $\{P_1, P_2, P_3\} \Rightarrow Conc$  is a **valid** sequent, i.e. that *Conc* is derivable from  $\{P_1, P_2, P_3\}$ .

Which of the following **MUST** be true? (The correct answer may include any number of these statements.)

1. *Conc* is a logical consequence of  $\{P_1, P_2, P_3\}$ .
2.  $\sim Conc$  is *not* a logical consequence of  $\{P_1, P_2, P_3\}$ .
3.  $\{P_1, P_2, P_3\}$  is a consistent set.
4.  $\{P_1, P_2, P_3\}$  is an inconsistent set.
5.  $\{P_1, P_2, P_3, Conc\}$  is an inconsistent set.
6.  $\{P_1, P_2, P_3, \sim Conc\}$  is an inconsistent set.
7.  $\{P_2, P_3, \sim Conc\}$  is an inconsistent set.
8.  $\{P_2, P_3, \sim Conc\}$  is a consistent set.
9.  $\{\sim P_1, P_2, P_3, Conc\}$  is an inconsistent set.
10.  $\{\sim P_1, \sim P_2, \sim P_3, Conc\}$  is a consistent set.
11.  $\sim P_1$  is a logical consequence of  $\{P_2, P_3, Conc\}$ .
12.  $\sim P_1$  is a logical consequence of  $\{P_2, P_3, \sim Conc\}$ .
13.  $\sim P_3$  is derivable in from  $\{P_1, P_2, \sim Conc\}$ .

14.  $P_3$  is derivable from  $\{P_1, P_2, Conc\}$ .
15.  $P_1 \rightarrow Conc$  is derivable from  $\{P_2, P_3\}$ .
16.  $P_1 \leftrightarrow Conc$  is derivable from  $\{P_2, P_3\}$ .
17.  $\sim Conc \rightarrow \sim P_3$  is derivable from  $\{P_1, P_2\}$ .
18.  $(P_1 \wedge P_2 \wedge P_3) \rightarrow Conc$  is derivable from  $\{\}$ .
19.  $(\sim P_1 \wedge \sim P_2 \wedge \sim P_3) \rightarrow \sim Conc$  is *not* derivable from  $\{\}$ .
20.  $P_1 \rightarrow (P_2 \rightarrow (P_3 \rightarrow Conc))$  is a logical truth.
21.  $\sim Conc \rightarrow (\sim P_1 \wedge \sim P_2 \wedge \sim P_3)$  is a logical truth.

### Part III What Must Be False?

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Which of the 21 sentences above **MUST** be false? (*Hint:* The answer is not just everything that wasn't correct in Part II.)

### Part IV Proofs by Induction

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Use structural induction to prove the following claims.

1. Every well-formed formula has the same number of left parentheses as right parentheses.
2. **Theorem (Substitution).** Let  $\varphi$  and  $\psi$  be logically equivalent formulas. We'll write this as  $\varphi \models \psi$ . Now let  $\chi$  be any formula, and let  $\chi'$  be the result of replacing zero or more occurrences of  $\varphi$  in  $\chi$  by  $\psi$ . Show that  $\chi \models \chi'$ .
3. **Duality.** Let  $\alpha$  be a wff whose only connective symbols are  $\wedge$ ,  $\vee$ , and  $\sim$ . Let  $\alpha^*$  be the result of interchanging every  $\wedge$  with  $\vee$  (and vice versa) and replacing each sentence symbol  $P$  by  $\sim P$ . Show that  $\alpha^* \models \sim \alpha$ .

*Remark:* It follows that if  $\alpha \equiv \beta$  then  $\alpha^* \models \beta^*$ .

4. Let  $\varphi$  and  $\psi$  be wffs whose *only* connectives are  $\sim$  and  $\leftrightarrow$ , and which have the same sentence letters (with the same number of occurrences of each) and the same number of  $\sim$  symbols and the same number of  $\leftrightarrow$  symbols. Show that  $\varphi \models \psi$ .

*Examples:*

- $\varphi = \sim(A \leftrightarrow B)$     and     $\psi = (A \leftrightarrow \sim B)$
- $\varphi = \sim(\sim A \leftrightarrow B) \leftrightarrow (\sim C \leftrightarrow A)$     and     $\psi = (\sim\sim\sim A \leftrightarrow (A \leftrightarrow (B \leftrightarrow C)))$

## Part V    Knights, Knaves, and Normals

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On a certain island there are **knights** who always tell the truth, **knaves** who always lie, and **normals** who sometimes tell the truth and sometimes lie. Knights have the highest rank, normals the middle rank, and knaves the lowest rank.

**A** says:    “*I am of a lower rank than B.*”

**B** says:    “*That’s not true!*”

Determine what rank each of A and B is, and whether what they say is true.

### Encoding into propositional logic.

Now let’s show how we can encode all of the information in this problem into propositional logic. Let’s use the following atomic sentences:

$$\begin{array}{ll}
 A_1 & = \text{Knight}(A) & B_1 & = \text{Knight}(B) \\
 A_2 & = \text{Knave}(A) & B_2 & = \text{Knave}(B) \\
 A_3 & = \text{Normal}(A) & B_3 & = \text{Normal}(B)
 \end{array}$$

Now using these atomic sentences, write each of the following as a formula of propositional logic.

1. A is exactly one of a knight, knave, or normal.
2. B is exactly one of a knight, knave, or normal.
3. If A is a knight, then A is of a lower rank than B.
4. If A is a knave, then it is *false* that A is of a lower rank than B.
5. If B is a knight, then it is *false* that A is of a lower rank than B.
6. If B is a knave, then A is of a lower rank than B.

*Note:* “A is of a lower rank than B” must itself be expressed in terms of the atomic sentences above. Think carefully about what combinations of  $A_i$  and  $B_j$  make the rank claim true.